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A new look at second-order equations and Lagrangian mechanics

W Sarlet†, F Cantrijn†§ and M Crampin‡

† Instituut voor Theoretische Mechanica, Rijksuniversiteit Gent, Krijgslaan 281-S9, B-9000 Gent, Belgium

‡ Faculty of Mathematics, The Open University, Walton Hall, Milton Keynes MK7 6AA, UK

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Abstract. To each second-order equation field Γ on a tangent bundle TM, we associate a particular subset $\mathfrak{X}^*_\Gamma$ of 1-forms on TM. Lagrangian systems then are characterised by the property that $\mathfrak{X}^*_\Gamma$ contains an element which is non-degenerate and exact. For general second-order equation fields Γ, we study type $(1, 1)$ tensor fields preserving both $\mathfrak{X}^*_\Gamma$ and a kind of dual set of vector fields $\mathfrak{X}_\Gamma$. Finally, we establish some theorems concerning $\mathfrak{X}^*_\Gamma$, which cover known results in Lagrangian mechanics.

1. Introduction

The last couple of years have shown a revival of interest in tangent bundle geometry and Lagrangian mechanics, which to some extent might be viewed as a reaction against the overemphasis on symplectic (cotangent bundle) geometry and Hamiltonian mechanics in modern literature. It is well known that the cotangent bundle T^*M of a differentiable manifold M carries a natural or intrinsic symplectic structure, which defines an isomorphism between vector fields and 1-forms on T^*M. A vector field then is Hamiltonian if it is mapped into an exact 1-form and different Hamiltonians yield different Hamiltonian vector fields on the same symplectic manifold. Concerning Lagrange's equations, it is customary to derive them too from a Hamiltonian vector field, this time on the tangent bundle TM. The situation there, however, is quite different: the Lagrangian L not only serves to define an associated exact 1-form, but is needed already to endow TM with a symplectic form; different Lagrangians accordingly give rise to different symplectic structures. If the Lagrangian formalism appears to be geometrically more complicated than the Hamiltonian one, it is largely due to this somewhat unsatisfactory feature.

However, it has been pointed out in several recent publications that the tangent bundle does itself possess some intrinsic structures, which play a significant role in the formulation of Lagrangian mechanics: see e.g. Klein (1962, 1974, 1983), Grifone (1972a, b), Crampin (1983a). The most important object in this respect is a type $(1, 1)$ tensor field S whose Nijenhuis tensor is zero and, when regarded as a map on $\mathfrak{X}(TM)$ (vector fields), is such that at each point its kernel coincides with its image. A general manifold endowed with an S having the above properties is called an

§ Research Associate at the National Fund for Scientific Research (Belgium).

integrable almost tangent manifold (see e.g. Clark and Bruckheimer 1960, Brickell and Clark 1974). An integrable almost tangent manifold can be regarded as generalising a tangent bundle in very much the same way as a symplectic manifold generalises a cotangent bundle (Crampin (1983b) has discussed the significance of this result in Lagrangian mechanics). It would perhaps be an exageration to conclude from this that S on TM is as fundamental for Lagrangian mechanics as the fundamental 1-form on T^*M is for Hamiltonian mechanics. Yet, it is equally true that the role of S has been underestimated in the conventional symplectic formulation of Lagrangian mechanics referred to at the beginning.

In the present paper we re-examine the description of Lagrange's equations of motion on TM. Our main point is that two distinct stages may be identified, to advantage, in the definition of a Lagrangian vector field. The first stage is reminiscent of the canonical isomorphism between $\mathfrak{X}(T^*M)$ and $\mathfrak{X}^*(T^*M)$ (the first stage in defining Hamiltonian vector fields); it is, however, a much weaker form of relation as it merely associates to every second-order equation field Γ on TM a set of 1-forms denoted by $\mathfrak{X}^*_\Gamma$. The intrinsic (1, 1) tensor field S on TM plays a dominant role in this association. The second stage bears even greater resemblance to the Hamiltonian picture, as it defines Γ to be a Lagrangian vector field when $\mathfrak{X}^*_\Gamma$ contains an exact 1-form. These matters are described in § 3 of the paper (§ 2 being devoted to some preliminary remarks on tangent bundle geometry).

As an immediate consequence of keeping these two stages separate, one is prompted to look again at the study of general second-order equation fields (not necessarily of Lagrangian type) on TM. Our principal idea here is that $\mathfrak{X}^*_\Gamma$ is an object worthy of more detailed study in its own right. To support this claim, we advance two types of arguments. First we show there exist some 'natural tools' for the study of $\mathfrak{X}^*_\Gamma$ in the form of (1, 1) tensor fields R preserving $\mathfrak{X}^*_\Gamma$ (§ 4). Next, we establish how a couple of well known results from Lagrangian mechanics, as for example Noether's theorem, already live at the first stage, i.e., there are theorems concerning $\mathfrak{X}^*_\Gamma$ which straightforwardly reduce to those known results when the system happens to be Lagrangian (§ 5). A number of related aspects and outlooks for further study are discussed in § 6.

2. Preliminaries

Let M be an n-dimensional differentiable manifold. On TM one can define an intrinsic (1, 1) tensor field S, which in terms of natural bundle coordinates (q^i, v^i) reads (with summation convention)

$$S = (\partial/\partial v^i)\otimes \mathrm{d}q^i. \tag{1}$$

In particular, S has constant rank n, it satisfies

$$S^2 = 0, \tag{2}$$

and its Nijenhuis tensor vanishes, a condition which may be most conveniently expressed for our purposes, in the form

$$\mathscr{L}_{S(X)}S = \mathscr{L}_X S\circ S \tag{3}$$

for all vector fields X on TM, where $\mathscr{L}_X$ denotes Lie derivation with respect to X. Note that from (2) it follows that for each vector field $X\in\mathfrak{X}(TM)$,

$$\mathscr{L}_X S\circ S = -S\circ\mathscr{L}_X S. \tag{4}$$

We recall that a type $(1, 1)$ tensor field can be regarded as a linear map on the module of vector fields or alternatively as a map on the module of 1-forms. In this paper we shall be concerned most frequently with the action of type $(1, 1)$ tensor fields on 1-forms. Thus we shall be dealing then with what in linear algebra would be the adjoint, or transpose, of the linear map corresponding to the action of the tensor on vectors. We make no notational distinction between the two. The reader is warned, however, that where our tensor fields are thought of as acting on 1-forms, there are minor differences between some of our formulae and those in papers by authors who adopt the opposite convention. In particular, in this paper, a composition of $(1, 1)$ tensor fields of the form $V = T \circ U$ will exclusively refer to the action on 1-forms: $V(\alpha) = T(U(\alpha))$ for all $\alpha \in \mathfrak{X}^*(TM)$. The action on vector fields then immediately follows by duality: $V(X) = U(T(X))$ for all $X \in \mathfrak{X}(TM)$. The commutator $T \circ U - U \circ T$ of type $(1, 1)$ tensor fields will be denoted by $[T, U]$.

In terms of the intrinsic tensor field S, a second-order equation field Γ on TM is fully characterised by the property $S(\Gamma) = \Delta$, where Δ is the dilation vector field $v^i \partial/\partial v^i$. In local coordinates Γ reads

$$\Gamma = v^i(\partial/\partial q^i) + \Lambda^i(q, v)\partial/\partial v^i.$$

It can be shown (Grifone 1972a) that the following intrinsic properties hold true

$$(\mathscr{L}_\Gamma S) \circ S = -S, \qquad S \circ (\mathscr{L}_\Gamma S) = S, \tag{5}$$

and that these further imply

$$(\mathscr{L}_\Gamma S)^2 = I, \tag{6}$$

where I is the identity tensor field (see also Crampin 1983a). For later use, we finally mention the following property, which follows immediately from (3) and (4).

Lemma 2.1. If $X \in \mathfrak{X}(TM)$ satisfies $S(X) = 0$ (i.e. X is vertical), then

$$S \circ \mathscr{L}_X S = 0. \tag{7}$$

3. Lagrangian vector fields

To each second-order equation field Γ on TM we now associate a set of 1-forms $\mathfrak{X}^*_\Gamma$ defined by,

$$\mathfrak{X}^*_\Gamma = \{\phi \in \mathfrak{X}^*(TM) \mid \mathscr{L}_\Gamma(S(\phi)) = \phi\}. \tag{8}$$

The elements of $\mathfrak{X}^*_\Gamma$ locally are of the form

$$\phi = \alpha_j(q, v)\, dv^j + \Gamma(\alpha_j)\, dq^j. \tag{9}$$

We note that all dv^j locally belong to $\mathfrak{X}^*_\Gamma$ and that $\mathfrak{X}^*_\Gamma$ is a real vector space but not a module over the ring of functions (it is, however, a module over the ring of 'constants of the motion', that is, functions f satisfying $\Gamma(f) = 0$).

Definition 3.1. $\phi \in \mathfrak{X}^*_\Gamma$ is called non-degenerate if $dS(\phi)$ is a symplectic form.

Locally this means that the matrix $(\partial\alpha_j/\partial v^i)$ is non-singular.

We now come to the characterisation of Lagrangian systems.

Definition 3.2

(i) A second-order equation field Γ is called a Lagrangian vector field if there exists a non-degenerate $\phi \in \mathfrak{X}_\Gamma^*$ which is exact: $\phi = \mathrm{d}L$ for some $\mathscr{C}^\infty$-function L.

(ii) Γ is called locally Lagrangian if there exists a non-degenerate $\phi \in \mathfrak{X}_\Gamma^*$ which is closed: $\mathrm{d}\phi = 0$.

Identification of the local expression (9) of ϕ with $\mathrm{d}L$ clearly shows that the differential equations corresponding to a Lagrangian vector field are indeed the Euler–Lagrange equations derived from L.

Remark. The condition $\mathscr{L}_\Gamma(S(\mathrm{d}L)) = \mathrm{d}L$, which characterises a Lagrangian vector field Γ, can immediately be rewritten in the form

$$i_\Gamma \,\mathrm{d}\theta_L = -\mathrm{d}E_L, \tag{10}$$

where $\theta_L = S(\mathrm{d}L)$ is the usual Poincaré–Cartan form and E_L is the energy function. So, obviously, nothing new is to be expected when the two stages of the definition of a Lagrangian vector field are joined together. As explained in the introduction, however, the idea is to insist on a distinction between these two stages. We therefore investigate in the next sections some properties of $\mathfrak{X}_\Gamma^*$ for arbitrary second-order equation fields.

4. Type (1, 1) tensor fields preserving $\mathfrak{X}_\Gamma^*$

In defining a new object like $\mathfrak{X}_\Gamma^*$, it is quite natural to wonder what kind of actions preserve that object. It is for instance readily seen that Lie derivation with respect to a vector field in general does not preserve $\mathfrak{X}_\Gamma^*$. However, $\mathfrak{X}_\Gamma^*$ is invariant under the action of suitable type (1, 1) tensor fields. In this section we will characterise tensor fields, which preserve $\mathfrak{X}_\Gamma^*$ and we will indicate a couple of ways of constructing such tensor fields out of other given objects.

Proposition 4.1. In order that a type (1, 1) tensor field R preserves $\mathfrak{X}_\Gamma^*$, it is sufficient that R commutes with S and satisfies $\mathscr{L}_\Gamma R \circ S = 0$.

Proof. Let $\phi \in \mathfrak{X}_\Gamma^*$, then

$$\begin{aligned}\mathscr{L}_\Gamma(S(R(\phi))) - R(\phi) &= \mathscr{L}_\Gamma(S(R(\phi))) - R(\mathscr{L}_\Gamma(S(\phi)))\\ &= \mathscr{L}_\Gamma([S, R](\phi)) + (\mathscr{L}_\Gamma R)(S(\phi)).\end{aligned}$$

Hence: $[S, R] = 0$ and $\mathscr{L}_\Gamma R \circ S = 0$ implies $R(\mathfrak{X}_\Gamma^*) \subset \mathfrak{X}_\Gamma^*$.

Local expression: starting from the general coordinate expression of a (1, 1) tensor field R on TM, it is straightforward to check that the requirements of proposition 4.1 imply that R has the following form,

$$R = a^j_k[(\partial/\partial q^j)\otimes \mathrm{d}q^k + (\partial/\partial v^j)\otimes \mathrm{d}v^k] + \Gamma(a^j_k)(\partial/\partial v^j)\otimes \mathrm{d}q^k. \tag{11}$$

Therefore, whenever we have at our disposal functions $a^j_k(q, v)$, which under an arbitrary change of bundle coordinates

$$\begin{pmatrix} q^j \\ v^j \end{pmatrix} \leftrightarrow \begin{pmatrix} Q^k = Q^k(q) \\ V^l = v^j(\partial Q^l/\partial q^j) \end{pmatrix} \tag{12}$$

transform as

$$a'^l_m = a^j_k(\partial Q^l/\partial q^j)\partial q^k/\partial Q^m, \tag{13}$$

we are sure that an R of the form (11) has all the right tensorial properties and thus defines a global object.

One obvious way of providing suitable functions a^j_k is as the coefficients of a given type (1, 1) tensor field A on the base manifold M. In such a case $\Gamma(a^j_k)$ is actually independent of the choice made for Γ, so that the resulting R will preserve $\mathfrak{X}^*_\Gamma$ for all second-order equation fields Γ. This is summarised in the following result.

Proposition 4.2. Let A be a type (1, 1) tensor field on M. There exists a unique type (1, 1) tensor field A^c on TM, characterised by the following properties,

(i) $A^c(\pi^*\alpha) = \pi^*(A(\alpha))$, $\forall \alpha \in \mathfrak{X}^*(M)$,

where $\pi: TM \to M$ denotes the tangent bundle projection,

(ii) $[A^c, S] = 0$

(iii) $\mathscr{L}_\Gamma A^c \circ S = 0$ for all second-order equation fields Γ.

The tensor field A^c is called the complete lift of A. Our construction, though different from that given by Yano and Ishihara (1973), has the same effect as theirs. Recall that the complete lift of a vector field $X = \xi^i(q)\partial/\partial q^i$ on M is given by $X^c = \xi^i\partial/\partial q^i + (\partial\xi^i/\partial q^j)v^j\partial/\partial v^i$ (for an intrinsic definition of X^c, see e.g. Crampin (1983a)). Apart from the similarity between the local expression for X^c and the formula (11) for A^c, the terminology concerning A^c is further justified by the fact that A^c maps complete lifts of vector fields into complete lifts.

Next, consider a general vector field Y on TM, whose local expression is given by $Y = \mu^i\partial/\partial q^i + \nu^i\partial/\partial v^i$. If we set

$$a^j_k = \partial\mu^j/\partial v^k, \tag{14}$$

it is easy to verify that a^j_k satisfies the transformation rule (13) under an arbitrary coordinate transformation of type (12). We thus discover that the prescriptions (14) uniquely associate an appropriate (1, 1) tensor field R to each given vector field Y, as is formalised in the following proposition.

Proposition 4.3. Let $Y \in \mathfrak{X}(TM)$ and Γ a second-order equation field be given. There exists a unique type (1, 1) tensor field R_Y on TM, characterised by the following properties,

(i) $S \circ (R_Y - \mathscr{L}_Y S) = 0$,

(ii) $[R_Y, S] = 0$,

(iii) $\mathscr{L}_\Gamma R_Y \circ S = 0$.

For the proof, it suffices to recognise that (i) intrinsically characterises the prescriptions (14).

Remarks

(1) If Y is projectable onto M, then $R_Y = 0$.

(2) If Y is itself a second-order equation field, then $R_Y = I$.

(3) An explicit expression for R_Y independent of coordinates may be derived as follows. From (ii), (i) can be written as $R_Y \circ S = S \circ \mathscr{L}_Y S$. Taking the Lie derivative with respect to Γ and using (iii) we obtain $R_Y \circ \mathscr{L}_\Gamma S = \mathscr{L}_\Gamma(S \circ \mathscr{L}_Y S)$. Composition with $\mathscr{L}_\Gamma S$, in view of (6), then gives $R_Y = \mathscr{L}_\Gamma(S \circ \mathscr{L}_Y S) \circ \mathscr{L}_\Gamma S$. Making repeated use of the

properties (4) and (5), this result may be written in the following alternative form, more convenient for later applications:

$$R_Y = \mathscr{L}_\Gamma S \circ \mathscr{L}_Y S + S \circ \mathscr{L}_{[\Gamma, Y]} S. \tag{15}$$

An intrinsic proof of the above proposition then consists in first verifying that (15) satisfies (i), (ii) and (iii) (this requires repeated use of the properties mentioned in § 2 again), and then checking that uniqueness follows from the easily proven property: if $R \circ S = 0$ and $\mathscr{L}_\Gamma R \circ S = 0$, then $R = 0$ (or indeed from the derivation of the formula for R_Y above).

(4) If $\mathscr{L}_Y S = 0$, then $R_Y = 0$.

It is of interest to introduce here a kind of dual of $\mathfrak{X}_\Gamma^*$, namely a subset of vector fields on *TM*, denoted by $\mathfrak{X}_\Gamma$ and defined by

$$\mathfrak{X}_\Gamma = \{ Y \in \mathfrak{X}(TM) \mid S([\Gamma, Y]) = 0 \}. \tag{16}$$

Locally, the elements of $\mathfrak{X}_\Gamma$ have the form

$$Y = \mu^i (\partial/\partial q^i) + \Gamma(\mu^i) \partial/\partial v^i. \tag{17}$$

They were studied under the name variation fields of Γ by Crampin (1983a). All symmetries of Γ belong to $\mathfrak{X}_\Gamma$ and are further (locally) characterised by the requirement

$$\Gamma\Gamma(\mu^i) = Y(\Lambda^i). \tag{18}$$

The duality between the formulae (9) and (17) is apparent; we have

$$\langle Y, \phi \rangle = \Gamma(\alpha_j \mu^j). \tag{19}$$

This relation in fact completely describes the duality between $\mathfrak{X}_\Gamma^*$ and $\mathfrak{X}_\Gamma$, so let us have a look at it in more intrinsic terms. For $\phi \in \mathfrak{X}_\Gamma^*$ and $Y \in \mathfrak{X}_\Gamma$, we have

$$\langle Y, \phi \rangle = \langle Y, \mathscr{L}_\Gamma(S(\phi)) \rangle + \langle S(\mathscr{L}_\Gamma Y), \phi \rangle,$$

the last term actually being zero in view of (16). This can be rewritten as,

$$\begin{aligned} \langle Y, \phi \rangle &= \langle Y, \mathscr{L}_\Gamma(S(\phi)) \rangle + \langle \mathscr{L}_\Gamma Y, S(\phi) \rangle \\ &= \mathscr{L}_\Gamma(\langle Y, S(\phi) \rangle), \end{aligned} \tag{20}$$

which is the same as (19). The above calculation, however, also shows us this: if (20) holds for all $\phi \in \mathfrak{X}_\Gamma^*$, then Y must belong to $\mathfrak{X}_\Gamma$ (dually, if (20) is satisfied for all $Y \in \mathfrak{X}_\Gamma$, then ϕ belongs to $\mathfrak{X}_\Gamma^*$). As an application of this result, consider again a (1, 1) tensor field R, satisfying the assumptions of proposition 4.1. Then, for all $\phi \in \mathfrak{X}_\Gamma^*$ and Y being an element of $\mathfrak{X}_\Gamma$, we have (knowing that $R(\phi) \in \mathfrak{X}_\Gamma^*$),

$$\begin{aligned} \langle R(Y), \phi \rangle &= \langle Y, R(\phi) \rangle = \mathscr{L}_\Gamma(\langle Y, S(R(\phi)) \rangle) \\ &= \mathscr{L}_\Gamma(\langle Y, R(S(\phi)) \rangle) = \mathscr{L}_\Gamma(\langle R(Y), S(\phi) \rangle), \end{aligned}$$

which implies that also $\mathfrak{X}_\Gamma$ is preserved under the mapping R, a property which of course could also be checked using the local expressions (11) and (17).

From (15), (16) and lemma 2.1, we immediately obtain: if Y belongs to $\mathfrak{X}_\Gamma$, then

$$R_Y = \mathscr{L}_\Gamma S \circ \mathscr{L}_Y S. \tag{21}$$

Note finally that $\mathfrak{X}_\Gamma$ contains the complete lifts of arbitrary vector fields on M.

Having established that there indeed exist rather natural tools for acting on $\mathfrak{X}_\Gamma^*$ (and $\mathfrak{X}_\Gamma$), we now turn to other aspects, which should give more weight to the present approach. In the next section, we will derive a number of results on $\mathfrak{X}_\Gamma^*$, of a quite general nature, which cover known theorems from Lagrangian mechanics in the following sense: they reduce to those theorems in case the ϕ of interest ($\in \mathfrak{X}_\Gamma^*$) happens to be exact.

5. Generalisation of known results in Lagrangian mechanics

Throughout this section, Γ will again be an arbitrary second-order equation field on TM.

Proposition 5.1. For each $\phi \in \mathfrak{X}_\Gamma^*$, we have

$$i_\Gamma(\phi - \mathrm{d}\langle\Delta, \phi\rangle) = 0. \qquad (22)$$

Proof.

$$\mathscr{L}_\Gamma(S(\phi)) = \phi \Leftrightarrow i_\Gamma\, \mathrm{d}S(\phi) + \mathrm{d}\langle\Gamma, S(\phi)\rangle = \phi$$

$$\Leftrightarrow i_\Gamma\, \mathrm{d}S(\phi) = \phi - \mathrm{d}\langle S(\Gamma), \phi\rangle.$$

The result now follows from $S(\Gamma) = \Delta$.

In case Γ is a Lagrangian vector field and $\phi = \mathrm{d}L$, it is clear that $\phi - \mathrm{d}\langle\Delta, \phi\rangle = -\mathrm{d}E_L$, so that (22) reduces to the usual conservation of energy. Of course, (22) cannot strictly be considered as a generalisation of the conservation of energy, as it does not represent a conservation law. The word generalisation is better justified for the following simple results.

Proposition 5.2. ('gauge freedom'). Let f be a given function on the base manifold M, then: $\phi \in \mathfrak{X}_\Gamma^* \Rightarrow \phi' = \phi + \mathrm{d}\Gamma(f) \in \mathfrak{X}_\Gamma^*$.

Proof. A simple coordinate calculation shows that $S(\mathrm{d}\Gamma(f)) = \mathrm{d}f$, from which it follows that $\mathscr{L}_\Gamma \mathrm{d}f$ belongs to $\mathfrak{X}_\Gamma^*$ for all $f \in \mathscr{C}^\infty(M)$.

Our next proposition deals with symmetries of Γ. Recall that a symmetry Y is a point symmetry if Y is the complete lift of some vector field on the base manifold (and satisfies $[Y, \Gamma] = 0$ of course). A point symmetry Y must satisfy $\mathscr{L}_Y S = 0$, since this holds for any complete lift; and in fact any symmetry Y such that $\mathscr{L}_Y S = 0$ is a point symmetry.

Proposition 5.3. Let Y be a point symmetry of Γ, then $\phi \in \mathfrak{X}_\Gamma^* \Rightarrow \mathscr{L}_Y\phi \in \mathfrak{X}_\Gamma^*$.

Proof. For $\phi \in \mathfrak{X}_\Gamma^*$ and $[Y, \Gamma] = 0$ we get,

$$\mathscr{L}_Y\phi = \mathscr{L}_Y\mathscr{L}_\Gamma(S(\phi)) = \mathscr{L}_\Gamma\mathscr{L}_Y(S(\phi))$$

$$= \mathscr{L}_\Gamma(\mathscr{L}_Y S(\phi)) + \mathscr{L}_\Gamma(S(\mathscr{L}_Y\phi)) = \mathscr{L}_\Gamma(S(\mathscr{L}_Y\phi)).$$

In case $\phi = \mathrm{d}L$, the statement of proposition 5.2 reduces to the well known gauge-freedom in Lagrangian mechanics, while proposition 5.3 then tells us that a point symmetry produces a new, equivalent (or sometimes only 'subordinate' (Marmo and Saletan 1977)) Lagrangian. We note in passing that the possible existence of equivalent Lagrangians fits quite well in our definition of a Lagrangian vector field, as it is not excluded that more than one element of $\mathfrak{X}^*_\Gamma$ could be exact (with more than just gauge difference).

The next result is less trivial and refers to symmetries which are not of point type. It is then no longer true that Lie differentiation preserves the whole of $\mathfrak{X}^*_\Gamma$; however, one may still manufacture another element of $\mathfrak{X}^*_\Gamma$ out of a given one, ϕ, by means of Lie differentiation provided that ϕ satisfies a certain condition.

Proposition 5.4

(i) If Y is a symmetry of Γ, and $\phi \in \mathfrak{X}^*_\Gamma$ satisfies

$$S(R_Y(\phi) - \mathrm{d}f) = 0, \tag{23}$$

for some $f \in \mathscr{C}^\infty(TM)$, then

$$\phi' = \mathscr{L}_Y\phi - \mathrm{d}\Gamma(f) \tag{24}$$

belongs to $\mathfrak{X}^*_\Gamma$.

(ii) Conversely, if $Y \in \mathfrak{X}_\Gamma$ and if there is a non-degenerate element ϕ of $\mathfrak{X}^*_\Gamma$ which satisfies (23) for an f such that ϕ' belongs to $\mathfrak{X}^*_\Gamma$, then Y is a symmetry of Γ.

Proof. From (24) we get

$$S(\phi') = \mathscr{L}_Y(S(\phi)) - (\mathscr{L}_Y S)(\phi) - S(\mathrm{d}\Gamma(f)).$$

A straightforward computation, using (8), then gives

$$\mathscr{L}_\Gamma(S(\phi')) - \phi' = \mathscr{L}_{[\Gamma, Y]}(S(\phi)) - \mathscr{L}_\Gamma(\mathscr{L}_Y S(\phi) + S(\mathrm{d}\Gamma(f)) - \mathrm{d}f). \tag{25}$$

From propositions 4.1 and 4.3, we know that $R_Y(\phi) \in \mathfrak{X}^*_\Gamma$. Therefore, taking the Lie derivative of (23) with respect to Γ, we obtain

$$\mathscr{L}_\Gamma S(\mathrm{d}f) + S(\mathrm{d}\Gamma(f)) = R_Y(\phi).$$

Operating on both sides with $\mathscr{L}_\Gamma S$ and taking account of (5) and (6), we see that

$$\mathrm{d}f - S(\mathrm{d}\Gamma(f)) = (\mathscr{L}_\Gamma S \circ R_Y)(\phi).$$

Now Y belongs to $\mathfrak{X}_\Gamma$, so that R_Y is given by (21) and thus,

$$\mathscr{L}_Y S(\phi) + S(\mathrm{d}\Gamma(f)) - \mathrm{d}f = 0. \tag{26}$$

With (26), the relation (25) reduces to

$$\begin{aligned}\mathscr{L}_\Gamma(S(\phi')) - \phi' &= i_{[\Gamma, Y]}\,\mathrm{d}S(\phi) + \mathrm{d}\langle[\Gamma, Y], S(\phi)\rangle \\ &= i_{[\Gamma, Y]}\,\mathrm{d}S(\phi).\end{aligned} \tag{27}$$

where the last step is a consequence of (16). Both parts of the proposition now immediately follow from (27).

It is instructive to look at the coordinate expression of the assumption (23), which for Y and ϕ respectively of the form (17) and (9) reads,

$$(\partial\mu^j/\partial v^i)\alpha_j = \partial f/\partial v^i. \tag{28}$$

We then see that part (i) of the previous proposition precisely covers a theorem by

Prince (1983), which states that (for the autonomous case) a symmetry Y of a Lagrangian system with Lagrangian L, satisfying (28) for some f (and with $\alpha_j = \partial L/\partial v^j$), produces a new (possibly equivalent) Lagrangian L', which is given by the formula $L' = Y(L) - \Gamma(f)$.

We finally arrive at a generalisation of Noether's theorem which, within the present framework, can be formulated as follows.

Proposition 5.5. Let Γ be a second-order equation field on TM and consider a non-degenerate $\phi \in \mathfrak{X}^*_\Gamma$. If $Y \in \mathfrak{X}(TM)$ satisfies,

(i) $\mathscr{L}_Y(S(\phi)) = \mathrm{d}f$, for some function f (29)

(ii) $i_Y(\delta - \mathrm{d}\langle\Delta, \phi\rangle) = 0$, (30)

then $F = f - \langle Y, S(\phi)\rangle$ is a first integral of Γ. Conversely, to each first integral F there corresponds a vector field Y satisfying (29) and (30).

Proof. From (29) we get

$$i_Y\,\mathrm{d}S(\phi) = \mathrm{d}(f - \langle Y, S(\phi)\rangle) = \mathrm{d}F.$$

Using the relation $i_\Gamma\,\mathrm{d}S(\phi) = \phi - \mathrm{d}\langle\Delta, \phi\rangle$ (see proposition 5.1) and (30) we then immediately find $\Gamma(F) = 0$. Conversely, since $\mathrm{d}S(\phi)$ is symplectic, each first integral F of Γ defines a unique vector field Y according to $i_Y\,\mathrm{d}S(\phi) = \mathrm{d}F$, from which (29) immediately follows with $f = F + \langle Y, S(\phi)\rangle$. Expressing $\Gamma(F) = 0$ further leads to (30).

Remarks

(1) In general, for a vector field Y satisfying (29) and (30), we have that

$$i_{[Y,\Gamma]}\,\mathrm{d}S(\phi) = \mathscr{L}_Y i_\Gamma\,\mathrm{d}S(\phi) = i_Y\,\mathrm{d}\phi \neq 0,$$

so that Y is not a symmetry of Γ. However, in case $\phi = \mathrm{d}L$, we do have $i_{[Y,\Gamma]}\,\mathrm{d}S(\phi) = 0$, which implies $[Y, \Gamma] = 0$ and so we are reduced to Noether's theorem for Lagrangian systems.

(2) In case ϕ is of the form $\phi = \mathrm{d}L + Q_i\,\mathrm{d}q^i$, Γ satisfies $i_\Gamma\,\mathrm{d}\theta_L = -\mathrm{d}E_L + Q_i\,\mathrm{d}q^i$, which shows that the corresponding differential equations are 'Lagrange's equations of the first kind' with 'non-conservative forces' Q_i. Proposition 5.5 then gives precisely the Noether theorem analogue for non-conservative systems as derived in Cantrijn (1982), in which the corresponding vector fields Y were indeed not symmetries of Γ.

6. Some further reflections

The preceding sections are meant to create new interest in the study of geometrical aspects of second-order differential equations. At the same time they contain many clues for further generalisations.

Let us first go back for a moment to the definition of Lagrangian vector fields in § 3. It is clear that the two steps of this definition could formally be maintained for vector fields on TM which are not of second-order type. One may wonder whether this would still have something to do with Lagrangian mechanics. So let X be an arbitrary vector field on TM with local representation

$$X = \rho^i(q, v)\partial/\partial q^i + \sigma^i(q, v)\partial/\partial v^i. \tag{31}$$

We again say that X is Lagrangian if there exists a non-degenerate, exact 1-form $\phi = \mathrm{d}L$, such that $\mathscr{L}_X(S(\phi)) = \phi$. This requirement locally means that

$$X(\partial L/\partial v^j) - \partial L/\partial q^j = -(\partial L/\partial v^k)\partial\rho^k/\partial q^j$$

and (32)

$$(\partial L/\partial v^k)\partial\rho^k/\partial v^j = \partial L/\partial v^j.$$

Considering the associated system of differential equations $\dot{q}^i = \rho^i(q, v)$, $\dot{v}^i = \sigma^i(q, v)$, we assume that $(\partial\rho^k/\partial v^j)$ is a non-singular matrix. Then, solving the first set of equations for the v^i, $v^i = \xi^i(q, \dot{q})$, we can define corresponding second-order differential equations as follows,

$$\ddot{q}^i = (\partial\rho^i/\partial q^k)(q, \xi(q, \dot{q}))\dot{q}^k + (\partial\rho^i/\partial v^k)(q, \xi(q, \dot{q}))\sigma^k(q, \xi(q, \dot{q}))$$
$$\equiv \Lambda^i(q, \dot{q}), \tag{33}$$

which defines the functions Λ^i. One can now verify the following rather remarkable feature: although a transformation of the form $(q, v) \leftrightarrow (q, \dot{q} = \rho(q, v))$ does not preserve the canonical structure of the fundamental (1, 1) tensor field S, it is correct to say that the second-order equation field Γ associated to (33) represents Euler–Lagrange equations corresponding to the Lagrangian $\bar{L}(q, \dot{q}) = L(q, \xi(q, \dot{q}))$. It may therefore be of interest to have a closer look at first-order equations (31) which satisfy the relations (32) for some L, even in the degenerate case where the matrix $(\partial\rho^k/\partial v^j)$ is singular.

The above remark solely relates to the Lagrangian content of the theory, i.e. it would not be meaningful to study $\mathfrak{X}^*_X$ for an arbitrary vector field X on TM. On the other hand, however, the interest of $\mathfrak{X}^*_\Gamma$ may extend beyond the strict requirement that Γ be of second-order type. As a matter of fact, when we return to the discussion of §§ 4 and 5, it becomes clear that the important properties of Γ we are using all the time are the relations (5), which in particular also imply (6). Hence, one could reconsider the whole theory, replacing second-order equation fields everywhere by vector fields Γ which satisfy (5), i.e. vector fields which, in Grifone's terminology, define a (non-homogeneous) connection $\mathscr{L}_\Gamma S$ on M (Grifone 1972a, definition I.14). One can easily check that, in local coordinates, a vector field Γ satisfying (5) will be of the form,

$$\Gamma = (v^i + h^i(q))\partial/\partial q^i + \Sigma^i(q, v)\partial/\partial v^i. \tag{34}$$

In terms of these vector fields one can again introduce Lagrangian systems according to definition 3.2 and the remark made about vector fields X as in (31) certainly remains valid for this more restrictive class (34). Moreover, one can now leave the framework of tangent bundle geometry for the more general almost tangent manifolds (see the introduction). A technical problem arising in Crampin's analysis of almost tangent structures (Crampin 1983b) was the fact that the definition of a second-order equation field there only makes sense after choosing a zero section for the local fibration defined by ImS. In view of the previous observation one can now avoid this difficulty by considering simply those vector fields on an almost tangent manifold, which satisfy the relations (5) with respect to the given tensor field S. All of this certainly merits further investigation.

To end this discussion, let us point out a couple of areas, which rather obviously call for an extension of the present considerations. First of all, we should work out a generalisation allowing for non-autonomous second-order equations. Secondly, we may contemplate a generalisation to differential equations of order higher than two with reference to so-called higher-order Lagrangian mechanics.

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